A KINDERGARTEN UNIT: CARDINALITY

by

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Abstract

This paper addresses the learning trajectory used in developing students' sense of cardinality and how this trajectory aids teachers in advancing students through their developmental progression of number sense. This paper also delineates the crucial aspect of subitizing in addition to one-to-one correspondence as viable methods for determining cardinality. An argument is made for the inclusion of subitizing into the learning trajectory and the Common Core State Standards. The use of manipulatives in teaching cardinality and misconceptions are also examined. The instructional framework known as Problem-Based Learning is discussed, and it is used in creating three cardinality lessons that correspond with the learning trajectory for cardinality and the Common Core State Standards.

Introduction

Children entering Kindergarten, come from various educational experiences. Some students will enter the classroom having been exposed to early numerical concepts; other students will enter with limited prior knowledge. It is crucial for student success in this first year of formal schooling that all students gain the knowledge of early numerical concepts. These early number concepts include four main topics: subitizing[1], verbal counting[2], enumerating objects[3], and cardinality (Clements & Sarama, 2014). Cardinality, the focus of this paper, is understanding how many total objects are in a group or set.

These early numerical concepts are taught beginning in preschool or at home, but they are addressed in Kindergarten to assure that students who did not attend preschool or receive at-home instruction gain the appropriate knowledge needed to succeed in mathematics. The early numerical concepts Clements and Sarama (2014) discuss build on the concepts stated in the Kindergarten Introduction of the Common Core State Standards (CCSS) (2010). According to the CCSS for the Kindergarten "Counting and Cardinality" section, by the end of the year Kindergarteners should be able to

> "…use numbers, including written numerals, to represent quantities and to solve quantitative problems, such as counting objects in a set; counting out a given number of objects; comparing sets or numerals; and modeling simple joining and separating situations with sets of objects…Students choose, combine, and apply effective strategies for answering quantitative questions, including quickly recognizing the

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[1] Subitizing refers to seeing or hearing a pattern and responding in some form (eye movement, pointing, or verbally) with the amount shown or heard

[2] Verbal counting is when children say the number names out loud.

[3] Enumerating objects indicates that children can point to an object and give it a number, thus connecting number names with values.

cardinalities of small sets of objects, counting and producing sets of given sizes, counting the number of objects in combined sets, or counting the number of objects that remain in a set after some are taken away" (Kindergarten Introduction, 2010).

Understanding these early numerical concepts, or counting and cardinality concepts, is essential for students' mathematical future (Confrey & Kazak, 2006). While reciting the counting sequence is understandably the first crucial component to math, cardinality is equally critical.

In this paper, I will discuss how learning trajectories play a role in the development of understanding cardinality and why cardinality is crucial to students' mathematical development. In addition, how teachers can support the development of cardinality will be addressed, including the use of manipulatives and early number sense misconceptions that arise. Furthermore, the pedagogical philosophy of problem-based learning and its relationship to mathematics achievement will be examined. This pedagogy is the driving philosophy behind the learning techniques in the cardinality lessons attached as appendices.

Learning Trajectories

Understanding why cardinality—knowing the last number stated is the amount of objects in a set without having to recount—is crucial to students' mathematical development can be explained partly through learning trajectories. Building on the works of Clements and Sarama (2014), Confrey and Kazak (2006), and Sztajn, Confrey, Wilson, and Edgington (2012), I define learning trajectories as frameworks that show the development of mathematical concepts. Teachers can use learning trajectories as a tool to understand the importance of certain concepts in a students' mathematical development. Children think about math in a natural developmental progression, and analogous to all mathematical concepts, cardinality follows a natural developmental progression of thought (Clements & Sarama, 2014). Teachers who understand the overall developmental progression and engage students in activities that will foster comprehension of mathematical concepts are more efficient and effective teachers (Clements & Sarama, 2014). Teachers can use learning trajectories as a tool for becoming adept at understanding children's developmental paths, for interacting with students to determine their knowledge base, and for providing appropriate activities for students (Sztajn et al., 2012).

In order to use learning trajectories as a tool, teachers need to be able to understand the way a learning trajectory is designed in order to glean the information provided within it. Learning trajectories have three components: "(a) a specific mathematical goal, (b) a path along which children develop to reach that goal, and (c) a set of instructional activities that help children move along that path" (Clements & Sarama, 2014, p. ix). Each goal is a big mathematical idea, or concept, which children learn by developing certain sub-concepts. The path within the learning trajectory is a compilation of sequential stages, in which children advance in their thinking process about the big ideas. The activities that are suggested for every stage reinforce the topics the child is learning and facilitate her grasp on the big idea. *Table 1* contains a fragment of Clements and Sarama's (2014) learning trajectory for counting. The trajectory is labeled by the big mathematical idea (i.e.-counting); column one titles the sub-concept in bold and lists what key knowledge is displayed during the respective stage, and column two lists suggested instructional activities beside each respective stage. Stages develop progressively and may overlap as students advance from one stage to the next. Clements and Sarama (2014) insert ages into the learning trajectory to show the development between stages, but I have removed them from the counting learning trajectory displayed, as students come into Kindergarten at different stages due to their exposure, or lack thereof, of content at home or preschool.

Table 1

Partial Learning Trajectory for Counting

Developmental Progression (stages)	Instructional Task
Pre-counter *verbal* Names some number words with no sequence.	Associate numbers as components of the counting sequence.
Reciter (10) *verbal* Verbally counts with separate words to ten.	Verbally count together both reciting and as you point to or move objects. Number magnets are one tool you could use to make the connection between the numbers they are saying and what its symbol and value is.

(Clements & Sarama, 2014, p. 36)

Once a teacher understands how a learning trajectory is designed, they can begin to use the information presented to aid students in moving through the developmental stages. Learning trajectories help teachers analyze students' work based on what stage they are functioning in because each stage delineates what knowledge students should be displaying about the sub-concept (Clements & Sarama, 2014; Sztajn et al., 2012). According to Clements and Sarama, teachers can use learning trajectories to

> "interpret what the child is doing and thinking...Similarly, when they interact with the child, these teachers also consider the instructional tasks and their own actions from the child's point of view so they can help the child develop the next level of thinking" (2014, p. 4).

As teachers assess where a student falls in the developmental progression, they can work with a student on the knowledge within the stage she is in and the knowledge displayed in the next stage to move the student along in her thought process.

As teachers interact with students and determine at what stage they are developing mathematical concepts, teachers can structure lesson plans more effectively to aid in moving students through the developmental progressions outlined in the learning trajectory. Teachers who use learning trajectories can narrow down specific knowledge that students need to work on in the developmental progression and support them with the instructional activities provided. Focusing lesson plans on these areas using the instructional activities appropriate for that level ensure that students are gaining the information they need at a level that is within their zone of proximal development (Sztajn et al., 2012; Vygotsky, 1987). Teachers who use research-based instructional activities in lessons increase student achievement because "they explicitly attend to conceptual understanding by addressing, discussing, and developing connections among concepts, facts, procedures, and processes…They challenge students to solve demanding mathematical problems, going beyond learning facts, helping them to learn to think mathematically" (Clements & Sarama, 2011, p. 970). Therefore, being well versed in the developmental progressions and research-based instructional activities of the learning trajectories allows teachers to create efficient and effective lesson plans for the benefit of her students' success (Sztajn et al., 2012).

Learning trajectories are an important resource for lesson planning because they correlate to the Common Core State Standards (CCSS). Learning trajectories help identify the big idea that a student is working on. Big mathematical ideas, or concepts, are the foundation of the CCSS that lessons build from: "when we wrote the Common Core we started by writing learning trajectories—at least the goals and developmental progressions. Thus, learning trajectories are at the core of Common Core" (Clements & Sarama, 2014, p. 6). Therefore, it makes sense to begin creating lessons for cardinality in Kindergarten by looking at the learning trajectory that corresponds to the "Counting and Cardinality" standards of CCSS.

Teachers can create lesson plans using the counting learning trajectory to cultivate an understanding of how cardinality is developed in a students' thought process, as well as the suggested activities that go along with this process. The counting learning trajectory is complex, involving three subtrajectories that all play a role in developing counting sub-concepts: verbal counting, object counting, and counting strategies (Clements & Sarama, 2014). For simplicity, these three subtrajectories have not been included in the abbreviated learning trajectory for counting. Table 2 displays an abbreviated version of the counting trajectory (displaying only stages and key knowledge developed) to show where cardinality falls within the counting learning trajectory.

Table 2

Abbreviated Learning Trajectory for Counting

Developmental Progression Title	Key Knowledge
Precounter	Names some number words with no sequence
Reciter	Names number words in a sequence, but not always in the correct order.
Corresponder	Keeps one-to-one correspondence between numbers and objects. May answer "how many?" by recounting.
Counter	Accurately counts arrangements of objects and answers "how many?" May be able to say number directly before or after a number.
Producer	Accurately counts out objects to an adult.
Counter and Producer	Counts and Counts out objects. *Has cardinality*. Begins to separate tens and ones.
Counter Backwards	Can count backwards by one
Counter from any number	Can count starting from numbers other than one.
Skip Counter	Can skip count forwards and backwards from any number
Counter of Place value	Understands the Base-Ten numeration system and place value concepts to solve

(Van de Walle, Karp, & Bay-Williams, 2013, p. 131)

In Table 2, the counting trajectory shows where cardinality falls in the developmental progression of thought children use in counting. As Table 2 insinuates, a child must understand cardinality before they can move on to higher counting competencies and eventually addition and subtraction problems. Students need an understanding of cardinality because arithmetic problems involve students taking groups of objects (whether physical or mentally imagined) and determining "how many objects they have after changes take place or as they compare quantities" (Van de Walle et al., 2013, p. 148). Students make changes to groups through adding or subtracting numbers to find the amount changed, or the starting amount, or the result (total after the change) (Van de Walle et al., 2013). If students do not comprehend cardinality, then their efficiency of solving arithmetic problems diminishes, as they are spending time reciting the counting sequence and recounting the counting sequence of groups without quantifying a group. Thus, it is vital for teachers to develop their students' understanding of cardinality, which is why Table 3 focuses specifically on the developmental progression children follow in the cardinality portion of the counting learning trajectory.

Table 3

Adapted Learning Trajectory for counting: Cardinality Stages

Developmental Progression
Corresponder Keeps one-to-one correspondence between counting words and objects (one word per object). May answer a "how many?" question by re-counting objects.
Counter (small numbers-5 and below) Accurately counts arrangements of objects and answers the "how many?" question with the last number counted. Begins to understand cardinality.
Counter (10 and below) Accurately counts arrangements of objects and answers the "how many?" question with the last number counted. May be able to tell the number before or after another number by counting forward by ones.
Producer (small numbers-5 and below) Counts out a certain number of objects for an adult. Recognizes that counting is relevant to situations where objects of a certain number must be placed.

Developmental Progression
Counter and Producer (10+) Accurately counts and counts out objects. Has explicit understanding of cardinality (how numbers tell how many). Can keep track of objects that have and have not been counted, even in different arrangements. Begins to separate tens and ones (leads to addition and subtraction).

(Clements & Sarama, 2014, p. 36-41)

Table 3 shows the five specific stages that generate an understanding of cardinality. The table does not include the first two stages because the first two stages involve reciting the number sequence and not quantifying objects like the rest of the stages that are listed. In order for children to be able to develop an understanding of cardinality, they must first learn how to recite numbers in sequential order, a fundamental process in mathematics, which occurs in the first three stages of the counting trajectory (Clements & Sarama, 2014). Once children can count sequentially, they begin to assign number values to objects, or one-to-one correspondence. As shown in Table 3, children who can give one-to-one correspondence are beginning to answer the question of "how many objects?" even if they need to recount; they are learning that a set of objects can be quantified, which is a precursor to cardinality (Morin & Samelson, 2015). After considerable exposure, children will not need to count the objects again, and they will say the last number as the answer to the "how many?" question. This is the stage Clements and Sarama (2014) label as counter-small numbers. Children expand on the "how many?" concept throughout the next three stages: to count larger groups of objects, to count out objects to adults, and to count and count out objects 10+, while keeping track of counted and uncounted objects. Once children can successfully complete these competencies, and understand that they can quantify a group of objects without recounting them, they have mastered cardinality. Children then go on to use cardinality in the subsequent stages and in higher cognitive tasks such as addition and subtraction (Clements & Sarama, 2014).

Cardinality

The intricacy of the developmental progression, illustrated in the learning trajectory of Table 3 demonstrates the complexity of cardinality, which Linnell and Fluck (2001) claim to be a multifaceted concept that is of higher importance to teach than assumed. As stated previously, cardinality is understanding how many total objects are in a group or set. Cardinality has multiple stages. In Clements and Sarama's (2014) cardinality portion of the counting learning trajectory, refer to Table 3, they lay out five specific stages that construct cardinality. However, within these stages, there are different levels of thought for students when responding to the question "how many."

Bermejo's (1996) research on cardinality acquisition indicated that one-to-one correspondence is not the only prerequisite for cardinality (1996). Two quantitative processes can determine cardinality: one-to-one correspondence and subitizing. Subitizing is different from one-to-one correspondence because it does not require a child to recite the counting sequence of the objects to know how many are in the set, as subitizing is instantly recognizing a known pattern with a cardinal number—one, two, three, etc. (Bermejo, 1996). Clements and Sarama (2014) make a similar argument claiming subitizing "is one of the main abilities…children should develop" (p. 9). Bermejo's study found that "first, some children did not know how to count, but they had cardinality, and second, 46% of the children used subitizing with three objects to answer correctly to cardinality, even in a counting situation" (Bermejo, 1996, p. 267). Thus, it is evident that some children do not use counting to determine cardinality, but rather use subitizing. This supports the claim that subitizing is an important competency to teach when introducing cardinality. Eimeren, MacMillan, and Ansari (2007) agree on the importance of subitizing in cardinality, as “building subitizing ability…supports object counting skills of correspondence and cardinality” (as cited in Clements & Sarama, 2014, p. 28) because “subitizing introduces basic ideas of cardinality—‘how many,’ ideas of ‘more’ and ‘less,’ ideas of parts and wholes and their relationships, beginning arithmetic, and in general, ideas of quantity” (Clements & Sarama, 2014, p. 10). Revealing that some children may use subitizing instead of one-to-one correspondence to determine the total number of objects in a set proves the important role subitizing plays in understanding cardinality.

It should be noted that subitizing is not a part of the counting learning trajectory that Clements and Sarama (2014) created or stated explicitly in the CCSS. However, subitizing may be inferred in the CCSS (2010) in the standards “Understand the relationship between numbers and quantities,” (KCC.B.4) and “… The number of objects is the same regardless of their arrangement …” (KCC.B.4.B). Subitizing should be inserted into the learning trajectory before one-to-one correspondence because some children may recognize patterns of numbers before they may enumerate them (Clements & Sarama, 2014). Furthermore, as important as subitizing has been proven to be, it should be explicitly stated in the CCSS amongst the cardinality objectives. For example, the new standard could read: Instantly recognize known patterns up to ten in a variety of arrangements to answer “how many”.

Subitizing has two forms: perceptual and conceptual (Clements, 1999). Perceptual subitizing is instantly recognizing how many are in a small group of objects. For example, seeing the pattern of four dots on a die and knowing there are four. Conceptual subitizing is using the patterns you can perceptually subitize and combining them together to determine how many are in a larger group of objects. For example, seeing two patterns of four dots on a domino and knowing that two patterns of four is a pattern of eight, so there are eight dots in all. In conceptual subitizing, children are adding known patterns together to create a new pattern. In using conceptual subitizing, students develop ideas about number sense and beginning arithmetic through recognizing the patterns as wholes and as a composite of parts (Clements & Sarama, 2014). For example, a student may see a pattern of five as a whole, or as a set of three and a set of two. This leads into thinking about numbers flexibly during addition and subtraction. Thus, subitizing allows students to "develop abstract number and arithmetic strategies" (Clements & Sarama, 2014, p. 10).

Many children will subitize to determine cardinality instead of, or before learning, one-to-one correspondence. In fact, children usually enter Kindergarten with the ability to subitize, and teachers unintentionally harm students' abilities by telling them to count the objects instead (Clements & Sarama, 2014). In doing so, children assume that one-to-one correspondence is a more appropriate method than subitizing, which is not true in all cases (Wright, Stanger, Cowper, and Dyson, 1994). For small groups of objects, subitizing is more efficient for determining cardinality, where as with larger groups, some conceptual subitizing may be able to be used, but one-to-one correspondence would be more efficient if the group cannot be subitized. Therefore, both one-to-one correspondence and subitizing are efficient ways of determining cardinality, and both strategies should be used when teaching (Clements & Sarama, 2014).

In order to achieve understanding of subitizing, one-to-one correspondence, and cardinality, students should be exposed to numerous experiences where groups of objects are enumerated (pointing to an object and giving it a number name, thus connecting number names with values), where they are verbally reciting the number sequence both individually and with others, and where they are asked the question "how many" of a set of objects (Clements & Sarama, 2014).

Teachers who consistently enumerate objects using one-to-one correspondence in everyday occurrences enable students "to build meaning for number words [i.e.-one, two, etc.] such as telling how many" (Clements & Sarama, 2014, p. 25). As children build this meaning, they begin to connect enumerating objects with the cardinality of said objects, moving from the corresponder stage to the counter stage (as described in Table 3). Furthermore, quantifying objects as a class contributes to individual success because students share reciting and enumerating strategies and the counting sequence is modeled correctly (Clements & Sarama, 2014; Bamberger & Schultz-Ferrell, 2010).

Quantifying objects addresses multiple competencies that students need to develop in order to solve addition and subtraction problems later in the math curriculum. Object counting addresses reciting the counting sequence forwards and backwards, skip counting, producing objects, quantifying object sets (cardinality), and determining a missing amount (how many are hidden?). While teachers effectively provide opportunities to enumerate and produce objects, they tend to neglect the question of "how many;" thus, ignoring the last two competencies —cardinality and determining a missing amount—developed by quantifying objects (Clements & Sarama, 2014).

Teachers can support the development of cardinality by consistently proposing the question "how many?" after students have counted objects. Children enjoy numbering objects and experiences at young ages; for instance, stairs climbed, times a ball is bounced, or cookies in a jar. These real-life experiences result in adults asking the "how many" question, making the connection between one-to-one correspondence and cardinality a more natural task (Clements & Sarama, 2014). More specifically, teachers can support cardinality through appropriate instructional tasks that are found in the third part of the cardinality portion of the learning trajectory. Table 4 displays suggested instructional activities appropriate for each developmental stage that was presented in Table 3. These tasks target specific cardinality competencies; such as, quantifying sets, producing objects, and determining a missing quantity.

Table 4

Abbreviated Learning Trajectory for Counting

Developmental Progression	Instructional Activities
Corresponder Keeps one-to-one correspondence between counting words and objects (one word per object). May answer a "how many?" question by re-counting objects.	*Count and Move:* Children count out loud as they move and touch a part of their body (touches head: one; touches nose: two, etc.).
Counter (small numbers-5 and below) Accurately counts arrangements of objects and answers the "how many?" question with the last number counted. Begins to understand cardinality.	*Cubes in Boxes:* Children count a small set of cubes (or other objects). Then put the cubes in a box and close the lid, hiding them. Ask the child how many cubes are hidden in the box. Play games such as *Sorry!® and Yahtzee®* where children move a certain number of spaces designated by a card, subitizing flash card, or die.
Counter (10 and below) Accurately counts arrangements of objects and answers the "how many?" question with the last number counted. May be able to tell the number before or after another number by counting forward by ones. (Leads to addition and subtraction)	*Counting Towers:* Have students build towers using up to ten objects. Ask how many objects they used. *How many Hidden?:* Tell students how many objects you have hidden and produce some of them. Have students tell you what amount is still hidden.
Producer (small numbers-5 and below) Counts out a certain number of objects for an adult. Recognizes that counting is relevant to situations where objects of a certain number must be placed.	*Count Motions:* During transitions in the classroom, ask students to complete or mimic a body motion (jump, clap, stomp, etc.) to a certain number while counting. At first, have them count out loud, then transition to having them count silently. Ask a child to place a certain number of objects (chairs, crackers, party hats, etc.) out for a group of people.

Developmental Progression	Instructional Activities
Counter and Producer (10+) Accurately counts and counts out objects. Has explicit understanding of cardinality (how numbers tell how many). Can keep track of objects that have and have not been counted, even in different arrangements.	Expand on activities in the previous two stages using higher numbers (*Counting Towers, Count Motions,* place objects out for people).

(Clements & Sarama, 2014, p. 36-41)

In Table 4, cardinality tasks in the counter stages focus on quantifying small and large sets of objects. Students can count blocks, Legos, people, animals, etc. After one-to-one correspondence, the teacher should ask "how many?" to aid students in understanding cardinality (Bamberger, Oberdorf, and Schultz-Ferrell, 2010). To challenge students, teachers can use the *Cubes in Boxes* instructional task. In this activity, the teacher shows the objects (e.g.-blocks) and then hides them away in a box before asking the student "how many?" objects are in the set. Teachers can even advance this hidden object exercise, using the *How Many Hidden* task to support development of determining a missing quantity. This is achieved through telling a student how many objects you have and then showing part of a set and keeping the rest hidden, encouraging students to mentally produce the objects needed to complete the number of objects in the set. This activity introduces the competencies needed to solve addition and subtraction problems as children are trying to find the missing amount using the result and the starting amount, which is one of three problem structures—finding the result, the amount changed, or the starting amount—for addition and subtraction problems (Van de Walle et al., 2013).

Other cardinality tasks in Table 4 include *Sorry!*® , *Yahtzee*®, *Count Motions,* and *Counting Towers* (Clements & Sarama, 2014). *Sorry!*® and *Yahtzee*® support cardinality because it requires children to count and move a certain number of spaces drawn or rolled. If teachers do not have access to these board games, any board game or dice activity will suffice. For example, instead of *Yahtzee*®, teachers could have students roll dice and add the numbers on the upright faces. Instead of playing *Sorry!*® , teachers can create a game board with spaces from start to finish and use a deck of cards or subitizing flash cards for students to draw and move. Subitizing flash cards are index cards with patterns of dots on them that can be subitized. *Counting Towers* and *Count Motions* support cardinality by requiring students to answer the "how many" question about their tower and produce a certain number of motions. *Counting Towers* and *Count Motions* have many variations, as any activity that a teacher can create or find that has students experiment with building, carrying, holding, or moving objects to determine or produce an amount of "how many?" fosters cardinality.

Cardinality is a concept that is rarely understood without the aid of physical objects or visual numbers. Understanding cardinality requires the action of giving a set of objects or a pattern of dots a number value. Thus, students need to be able to see and/or manipulate a group of objects. These objects are called manipulatives in mathematics. Manipulatives aid students in comprehension of content material through the ability to "manipulate", or move, objects around. Students learning cardinality point to objects or a pattern of dots, or move the physical objects, as they count them. These actions help students enumerate and quantify sets of objects. Thus, specific manipulative use in cardinality is important to understand.

Manipulatives

Manipulatives used in cardinality tasks enhance a students' understanding of the mathematical concept being explored. Understanding cardinality rests in part on the use of manipulatives, as all the stages require use of objects to count and/or produce. Morin and Samelson state that: manipulatives are an "instructional power…to convey important mathematical concepts" (2015, p. 364). Manipulatives, in a technical sense, range from concrete objects, to semiconcrete objects, and finally to abstract numerical symbols (Morin & Samelson, 2015). When beginning to learn a mathematical concept, students will manipulate a concrete object, such as blocks, dual-colored counters, or uni-fix cubes. As they grow comfortable with the concept, they gradually move from working with concrete manipulatives to semiconcrete manipulatives. Semiconcrete manipulatives are visual representations of manipulatives that students draw out, such as tally marks, dots, or pictures of uni-fix cubes. Students will then begin to use abstract manipulatives—formal number symbols (1, 2, etc.)—to represent the amount of a set.

When initially working to determine cardinality, students are enumerating concrete manipulatives and answering the question "how many." Students recite the number sequence and quantify any set of concrete objects presented to them in instructional activities. For example, in *Counting Towers*, students are using coins or blocks to create a tower and then answer "how many?" coins or blocks they used to create the tower. Students should use concrete manipulatives when beginning cardinality so that they can move them around and keep track of which objects have or have not been counted (Bamberger et al., 2010). As students practice cardinality and move towards simple addition and subtraction problems they begin to use cardinality with semiconcrete sets of objects. Students quantify sets of semiconcrete objects using pictures on worksheets, as well as drawing out sets of objects, in order to answer the question "how many." When students become comfortable using written numerals to represent the number of a group over a semiconcrete object, they have moved on to using abstract manipulatives. Thus, whenever students are working with cardinality, they are using a manipulative to quantify a group of objects.

Considering the critical role that manipulatives play in understanding cardinality, it is crucial for teachers to recognize the misconceptions that can arise with the use of manipulatives. Cardinality is one of the beginning encounters students have where manipulatives (i.e.-objects) are given a numerical value. Thus, students working with concrete manipulatives—blocks for example—have to be aware that the block represents the physical block and a quantity, or a number (Morin & Samelson, 2015). Further, when students are enumerating concrete objects, it is encouraged for them to point to or move to the side each object as they number them. These actions support accurate one-to-one correspondence, but they can also lead to errors. According to Morin and Samelson (2015), students who point to objects or move them for one-to-one correspondence could: point to an object twice, misplace an object into the wrong pile of counted or not, forget which group has been counted or not, and say the incorrect oral count to the object pointed to or moved. Thus, teachers should model enumerating concrete objects through pointing and moving the objects to model both the accurate counting sequence and accurate one-to-one correspondence.

In addition, it is important for teachers to provide congruent manipulative displays for students; meaning, objects the students are quantifying should all be the same. For example, if a teacher provides a set of objects that has blocks, counters, and counting bears, a student may not know which 'set' of objects to count. To the student each type of object is a group or set, instead of all of the types of objects being one group to count. Therefore, teachers should use only one type of object in the group needing to be quantified. Another example would be if a student is asked to compare which set of objects is larger—meaning quantitatively—but one set of objects is physically larger or takes up more space than the other set. When one set of objects physically takes up more space than the other set, a student may answer that the physically larger set is the larger set quantitatively, regardless of the actually number of objects in each set (Morin & Samelson, 2015).

Further, similar to using congruent manipulatives, a teacher must be cognizant of using situation-appropriate manipulatives. For example, if a student is asked to complete the *Counting Towers* activity, but the teacher provides the student with balls instead of blocks, the student may be unable to complete the activity because it is difficult to stack balls on top of each other and not have them fall. In another example, if a teacher presents a story problem requiring students to determine how many apples are in the basket, the teacher should provide a representation (e.g.-physical, pictoral, or stickers) of matching manipulatives—apples and a basket (Morin & Samelson, 2015). Providing incongruent and inappropriate manipulatives leads to student misconceptions about what objects to assign cardinality, which leads to errors in their accuracy of assigning cardinality to groups of objects.

Additional errors can arise when using one-to-one correspondence to determine cardinality for students who have difficulty with the counting sequence. Students who have difficulty with the counting sequence may struggle using one-to-one correspondence to determine cardinality because they may incorrectly enumerate the set of objects. These errors derive from misconceptions about the number names and counting sequence, which affects not only students' understanding of cardinality, but their understanding of all mathematical concepts because knowing the counting sequence is the basis of all mathematical computation. Often times, students not only have misconceptions about the use of manipulatives, but also about the counting sequence, which can affect their learning with manipulatives and of cardinality.

Misconceptions about the counting sequence

Misconceptions about the counting sequence may arise for some students when they are beginning to learn how to recite the number names sequentially. Students have misconceptions about assigning number names to the first ten digits and when reciting the teen numbers, which will occur with larger sets of objects. A student who has difficulty with number naming may believe that each number from one to ten must be one syllable, and therefore the number seven (which is the only single digit that has two syllables) is two different numbers: se and ven (Bamberger et al., 2010). If students have this misconception, they will miscount the objects in a group of seven or more, generating an incorrect cardinality answer. These misconceptions can be rectified through teachers modeling the accurate counting sequence and accurate correspondence.

Moreover, many students struggle with reciting the teen numbers, as they do not follow a pattern like numbers 1-10 and 20 and beyond (Bamberger et al., 2010). Children will come up with a pattern that makes sense to them when reciting teens, until the correct number names and sequence is repeatedly modeled to them. Students typically struggle most with thirteen and fifteen, as they are based on the ordinal numbers third and fifth, instead of the cardinal numbers three and five. Students also struggle with eleven and twelve, since they do not follow the pattern of either the cardinal (i.e.-one, two, etc.) or the ordinal (i.e.-first, second, etc.) numbers. The rest of the teen numbers begin with the cardinal number (i.e.-fourteen, sixteen, etc.), and students master these fairly quickly (Bamberger et al., 2010).

Bamberger et al. (2010) provide an example of a student who miscounts the teens. The student counts the teens as “one-teen, two-teen, three-teen, four-teen, five-teen, six-teen, seven-teen, eight-teen, and nine-teen” (p. 3). This student “overgeneralized the counting pattern and put each number word (from one to nine) in front of the teen numbers" (Bamberger et al., 2010, p. 4). This is not an uncommon method to reciting the teen numbers. In fact, this is similar to how the teen numbers are translated in Japanese, Chinese, and Welsh. The teen numbers in these languages support the base-ten system, naming them as: 'ten-one,' 'ten-two,' 'ten-three,' etc. (Ryan & Williams, 2007, p. 54). Thinking of teen numbers in this way is an advantage to being able to unitize—thinking of ten flexibly as a unit of ten and as 10 ones—later addition, subtraction, multiplication, and division (Taylor, Breck, & Aljets, 2004). Unfortunately, America's system for reciting the teen numbers is more confusing. The student in Bamberger et al., and any student who has trouble with reciting teen numbers in the American system, can be corrected through repetition of exposure and hearing the teen numbers being recited correctly. This can be done through class or individual counting sequence chants, counting sequence songs, reading counting sequence books, or reciting the counting sequence out loud (Bamberger et al., 2010; Bamberger & Schultz-Ferrell, 2010). Students who struggle with the counting sequence need the modeling by the teacher and repetitive exposure to correct their mistakes so that when they assign cardinality, they are both accurately numbering the objects and correctly naming how many are in the set.

Teacher modeling and numerous experiences with the material are two ways to correct misconceptions. Another important way to address misconceptions is to allow students to work with manipulatives prior to a standard algorithm so that students understand the conceptual mathematics before being taught a "correct" standard algorithm. Standard algorithms may be confusing to a student who has not had the mathematical background of why it works, which can be understood through working with manipulatives. Thus, it is important how the numerous experiences are provided. Students need interactive, engaging, and relatable experiences with the material so that they can grasp the concepts in their own terms. Instead of passive learning where the teachers pass the information onto the students, or traditional teaching, students need to experience the material for themselves and learn about the concepts with the teacher guiding or facilitating them in their learning. This type of learning is imperative for the concept of cardinality because students need to explore and practice the mathematical concepts, such as subitizing and one-to-one correspondence, involved in understanding cardinality. The teacher can guide the student to the answer, but telling her the total of a group of objects does not help her understand why the group is that number; whereas, if the student counts out the objects she can see the reason the group of objects is labeled with that number name. The student-centered, active, and engaging learning process is a part of the pedagogical framework of problem-based learning.

Problem-Based Learning

Problem-based learning (PBL) is a pedagogical technique that involves experiential learning (exploring, explaining, resolving) through real-life problems (Barrows, 2000; Hmelo-Silver, 2004; Torp & Sage, 2002). PBL is one technique in the family of the student-centered pedagogical frameworks in which students construct knowledge and create strategies by working on meaningful tasks. In this framework, the teacher acts as a facilitator to guide students in their learning and thinking. Other teaching techniques in this framework are inquiry-based learning, project-based learning, conceptual model-based problem-solving (COMPS), and active learning (Hmelo-Silver, 2004). Henceforth, when PBL is discussed, all of these techniques are encompassed, as they vary only in the type of product or task assigned and not the pedagogical framework (Hmelo-Silver, 2004).

Problem-based learning is an effective method in the classroom for both teachers and students (Wood, Cobb, & Yackel, 1991). The main benefit that the PBL pedagogy provides for teachers is that it has changed their perceptions on the meaning of instruction. In Wood et al.'s (1991) landmark study, the teacher they studied came to realize that by implementing a PBL teaching approach, she was able to understand the way her students thought about mathematics, which in turn improved her ability to help her students learn better than when she taught traditionally. A PBL approach allows teachers to better understand their students thought processes because of the nature of the classroom communication. In a PBL classroom, students are encouraged to explore, discuss, and use different strategies (Hmelo-Silver, 2004). The discussion aspect is critical for students and teachers to understand the thought processes happening. Furthermore, this facet of PBL is what allows Kindergarten teachers to assess their students' place in the cardinality development of progression in Clements and Sarama's (2014) counting learning trajectory. In turn, as previously discussed, teachers can better facilitate their students' learning by targeting their instructional needs.

For students, problem-based learning provides multiple benefits that lead to higher achievement. Primarily, PBL supports autonomy, and higher autonomy leads to high cognitive engagement, which leads to higher success rates (Rotgans & Schmidt, 2011). PBL supports autonomy in students because it creates a supportive environment for students to explore and learn the content (Hmelo-Silver, 2004). In math, this is particularly important because it allows students to derive strategies instead of being told and shown a 'correct' way to solve a problem. Students who engage in the mathematical content in this depth increase their content knowledge and "doing with understanding" because they are trying to understand the problem and not just follow the procedure (Barron, Schwartz, Vye, Moore, Petrosino, Zech, & The Cognition and Technology Group at Vanderbilt, 1998, p. 276; Meyer, Spencer, & Turner, 1997; Hmelo-Silver, 2004.

Moreover, problem-based learning boosts student attitudes towards math and their motivation to learn math. Students who were taught with the PBL approach were significantly more motivated to learn (Stipek, Salmon, Givvin, Kazemi, Saxe, & MacGyvers, 1998), had higher positive attitudes towards math (Rukavina, Zuvic-Butorac, Ledic, Milotic, and Jurdana-Sepic, 2012), and were more active in seeking mathematical challenges and taking mathematical risks (Meyer et al., 1997). PBL is effective in supporting these areas because it involves peer collaboration and mastery learning (Meyer et al., 1997; Rotgans et al., 2011; Stipek et al., 1998). Peer collaboration involves students in sharing problem-solving strategies, which expands content knowledge and produces comradeship between peers instead of competition (Rotgans et al., 2011; Stipek et al., 1998). Peer collaboration plays a role in learning cardinality because students share strategies for learning the counting sequence, how and when to subitizing or use one-to-one correspondence, and when moving from concrete to abstract manipulatives. Mastery learning is an environment in which failure is deemed as part of the learning process and therefore a positive action; this allows students to feel comfortable taking risks—derive their own strategies—and face challenging problems (Meyer et al., 1997). This helps students learning cardinality, particularly when determining a missing quantity, because it allows students to attempt the problem without consequence.

Lastly, problem-based learning allows traditionally underrepresented students in mathematics (low SES, females, and students with disabilities) to exhibit mathematically promising characteristics as outlined by Sheffield (2003). These include: a) mathematical frame of mind, b) mathematical formalization and generalization, c) mathematical creativity, and d) mathematical curiosity and perseverance (Trinter, Moon, & Brighton, 2015). The PBL context in conjunction with Sheffield's (2003) characteristics levels the playing field, particularly in Kindergarten, for students who did not benefit from preschool or at-home instruction in early numerical concepts, such as cardinality. The enriched learning environment of conceptual thinking, active learning, and peer collaboration that PBL offers are what support the four characteristics because these methods allow students to construct their strategies, answers, and knowledge (Trinter et al., 2015; Sheffield, 2003; Wood et al., 1991). Students who construct their own learning tend to have a deep understanding of the content, which leads to higher achievement (Wood et al., 1991).

Conclusion

The positive impact problem-based learning has on mathematics achievement is a benefit to teacher pedagogy in the subject and proves practical for use in the design of the affixed cardinality lessons. When teaching cardinality—understanding how many total objects are in a group or set—teachers need to include teaching early number concepts such as verbal counting, enumerating, subitizing, and one-to-one correspondence, as these all contribute to understanding cardinality. It is important that teachers incorporate the use of manipulatives in their lessons to aid and engage students. Further, they must be aware of the errors and misconceptions that can arise when working with manipulatives and learning the number sequence so that they are prepared to correct these errors and misconceptions. Moreover, teachers can use learning trajectories to aid them in understanding and advancing students' developmental progression of number sense and, in particular, cardinality. These learning trajectories provide suggestions for instructional activities that students can engage in to foster understanding of cardinality.

The learning trajectory for cardinality is beneficial for teachers in all of the ways aforementioned, but it should include subitizing. Subitizing is just as crucial to answering the question “how many” as is one-to-one correspondence; in fact, many children naturally subitize to determine cardinality. Subitizing allows students to “develop abstract number and arithmetic strategies” through recognizing the patterns as wholes and as a composite of parts (Clements & Sarama, 2014, p. 10). Granted, Clements and Sarama (2014) created a separate learning trajectory for subitizing because of its importance in developing number sense, but to ignore its role in the cardinality learning trajectory prevents teachers from being able to successfully teach their students and advance them in their development of number sense. In the learning trajectory for cardinality, subitizing should be inserted before one-to-one correspondence, as some children may recognize patterns before they may enumerate them. Furthermore, the CCSS does not include subitizing, which encourages teachers to skip over this key knowledge because it is not deemed important. As previously suggested, subitizing should be included between standards KCC.B.4 and K.CC.B.4.B, reading as: Instantly recognize known patterns up to ten in a variety of arrangements to answer “how many.” Inserting this standard for subitizing into the common core ensures that subitizing is taught and, more importantly, connected to cardinality.

The attached lesson plans follow a developmental progression in accordance with my revised cardinality learning trajectory—adding subitizer before the corresponder stage. The first lesson involves students in answering “how many” by instantly recognizing patterns on dot plates. The discussion following this activity requires students to think about how they could use subitizing to determine cardinality and whether it helps them answer the question “how many” easier or not, depending on the number. This lesson explicitly teaches subitizing as connected to cardinality to students, as the research suggests should occur. The second lesson connects to the corresponder stage in the learning trajectory, the next stage after subitizing (as inserted). The second lesson has students use one-to-one correspondence to count pom-poms into empty water bottles according to the number on the bottle; upon finishing they will answer "how many" are in each bottle on a worksheet. This lesson has two purposes: to identify students' ability to produce an amount, and to identify the ability to determine cardinality using one-to-one correspondence; both of which are in the CCSS standards K.CC.B.4, B.4A, and the second half of B.5. Finally, the third lesson builds upon the first two by entering into the counter of small numbers stage. In this lesson, students participate in playing a board game that requires them to either use one-to-one correspondence or subitizing to determine how many spaces to move. Thus, students are using both mediums to determine cardinality, as well as being able to produce the amount by moving their game piece on the board. These tasks connect to the CCSS standards: K.CC.B.4, B.4B, B.4C, and B.5.

These three cardinality lesson plans have been developed using the problem-based mathematics planning format Launch, Explore, Summarize, the learning trajectory for cardinality suggested by Clements and Sarama (2014), and the Common Core State Standards for Kindergarten. It is my hope that after reading how critical cardinality is to teach, as well as the benefits of using a problem-based learning pedagogy, that teachers will have a stronger focus on cardinality in their classrooms and use the techniques and instructional activities provided in this paper, as well as the attached lessons as resources for their mathematics curriculum.

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APPENDIX A

Research-based Instructional Design Practices Specific to Mathematics

—*Launch, Explore, Summarize Lesson Plan Format*—

1. Description of the Lesson

    - ❖ Subject: Math-Cardinality
    - ❖ Grade: Kindergarten
    - ❖ Timing (How long will the lesson take?): 30 min

    - ❖ Reference for lesson: Van De Walle, J., Karp, K., & Bay-Williams, J. (2013). Developing early number concepts and number sense. In Canton, K. (Ed.), *Elementary and middle school mathematics: Teaching developmentally* (8^{th} ed., pp. 129-130). New Jersey: Pearson.

2. Learning Goals and Objectives

    - ❖ What is the essential content of the instruction? (What are the big mathematical ideas of the investigation?)
        - ➢ Students will practice cardinality by answering "how many?" using subitizing.
    - ❖ CCSS Standard (What do I want kids to know when this investigation is finished?)
        - ➢ From thesis, pg. 11, paragraph 2: Instantly recognize known patterns up to ten in a variety of arrangements to answer "how many."

    - ❖ What are my specific and measurable objectives for the lesson?
        - ➢ Students will answer "How many?" by subitizing up to 10

    - ❖ Consider student background, knowledge and experience. What prior knowledge do the students have concerning this topic?
        - ➢ Students need to be able to recite the number sequence (count) up to 10
    - ❖ What mathematical vocabulary does this investigation bring out?
        - ➢ Number- a value, quantity, or amount expressed by a word, symbol, or figure
        - ➢ Number symbol- a symbolic representation of a number (tallies, dots, etc.)
        - ➢ Manipulative terms: Dot Plate, patterns, dots, die
    - ❖ What misconceptions might arise?
        - ➢ May try to use one-to-one correspondence in the limited amount of time they are shown the arrangement.
    - ❖ Curriculum adaptations and instructional modifications: (modifications to meet the needs of diverse populations and students with various ability levels)
        - ➢ Larger images for visually impaired students

- ➢ Larger manipulatives for students with delayed fine motor skills.

3. Resources, Materials, and Preparation for Instruction (What facilities, resources, and tools will be needed and how are they to be used?)

- ❖ Collaborators (Other teachers, outside visitors, or technical staff):
  - ➢ Assistant teacher, if there is one, to help monitor stations and assist students
- ❖ Technological tools for the teacher (demonstration):
  - ➢ May use a doc camera if you wish to show dot plate, or you may just hold it up in front of students—make sure they all can see the plate.
- ❖ Technological tools for the student:
  - ➢ N/A
- ❖ Manipulatives:
  - ➢ Dot Plate (paper plate with office supply colored dots formed in patterns on the plates); die for launch.
- ❖ Handouts and recording sheets (include a copy):
  - ➢ Personal recording: Keep track of student's responses to see how many they are getting correct and which patterns/numbers they are having trouble with.

Instructional Method and Procedures

a. Launch (5-10 minutes)
This is when you give students the information they need to do the lesson and solve the problem or task. You want to give students enough information so that they can do the lesson — but don't give too much away at this point. Unless you have to do a mini-lesson to refresh students' memories about a certain concept, <u>avoid direct instruction</u>.

Teacher Considerations (Before)	Description of Learning Activities	Anticipated Student Responses	Teacher Guidance (During)
• I will engage students' prior knowledge by having them count along with the Youtube video: https://www.youtube.com/watch?v=6RfIKqkvHTYThen, I will ask how many students have played with a die before and briefly discuss the use of subitizing (using an open ended	We're all going to count along with the video! Get your counting voices ready! I want to hear you loud and proud. Who has played with a die before, raise your hand (hold one up)?	Student(s) count in standard order along with the video, loudly and clearly. Students: raise hand if they've	Positive reinforcement for those who are participating. Do the video with the students! Acknowledge and validate counting the dots as a

format so as not to detract from the summarize section).	How do you figure out what number you rolled? (call on students to answer)	played with a die before. For 'how many?' may answer with: counted the dots, I just know	viable way to determine how many you rolled. If students don't bring up just knowing it, bring it up yourself. Then say: "That's what we're going to play with today… just knowing the number of dots we see without having to count them."

b. Explore (5-10 minutes)
This is where students work individually or in small groups to solve the problem. This is their chance "to get messy with the math."

Teacher Considerations (Before)	Description of Learning Activities	Anticipated Student Responses	Teacher Guidance (During)
• Supplies: • Advantage: Learning to subitize aids in counting on and learning combinations of numbers (Van de Walle, 2013). This particular activity allows the teacher to control how long students see the pattern, making it harder for them to enumerate and	Directions: We're going to play a game now. It's called Dot Plate Flash. I'm going to flash a dot plate at you for five seconds. Let's all count that together saying Mississippi in between each number so you know how long that you'll have. Ready? 1 Mississippi, 2 Mississippi, 3 Mississippi, 4 Mississippi, 5 Mississippi.	1, 2, 3, 4, 5, 6, 7, 8, 9. 10	If incorrect—or if someone disagrees, have and/or help them count the dots and look for patterns to help them count the dots faster. Also, you can show them multiple objects that have this pattern to show that the pattern is three. The more comfortable students become with the

enables them to learn subitizing. • The dots on the plates for beginners should be in easy to recognize patterns (straight lines, patterns they see on a typical die). • As their confidence builds (both in this lesson and subsequent lessons), introduce higher numbers and harder patterns (rectangular arrays, circles, scattered arrangements).	Everyone feel how much time 5 seconds is? Okay. So I'll flash a plate to you for 5 seconds, and in that time I want you to figure out how many dots are on the plate. Try to look for patterns to help you. When I hide the plate I'm going to ask 'How many dots were there?' and you'll answer altogether okay? Let's do one for practice. Doing: Flash dot plates for 5 seconds each, and then ask 'How many dots were there?' after each plate.		activity, and for more of a challenge, you can start lowering the time for showing the cards (3 seconds instead of 5, then 2, then 1) and/or making the patterns on the dots harder (see teacher considerations as well).

c. Summarize (10 minutes)

The purpose of the Summarize section is to bring groups back together and have students explain their solutions while assessing how students are progressing towards the goals of the lesson. The teacher's role is to guide students to the big ideas, to make sure that they have nailed the mathematics. Use the discussion to help you determine whether additional teaching and/or additional exploration by students is needed before they go on to the next lessons.

Teacher Considerations (Before)	Description of Learning Activities	Anticipated Student Responses	Teacher Guidance (During)
• Discuss which patterns were easier or harder	Which numbers were easier to just see? (show an example of plates and ask which	1, 2, 3, 4, maybe 5 and 6	Why were those ones easier? Did you count them one by one, and that still

• Bring up the definition for the term subitizing—maybe even the term, but it's not necessary for them to know the term. • Discuss whether it would be faster to enumerate or subitize when counting objects in a group.	was easier or harder) Which numbers were harder to just see? Which patterns were easier to recognize? Which patterns were harder to recognize? Were bigger numbers easier to figure out if it had two of a smaller pattern (five and five for 10)?	7, 8, 9, 10, maybe 5 and 6 1-6 because we see these on dice a lot of times (we use dice in other lessons). 7-10 Yeah, because we could use the smaller patterns we knew better and put them together, like I know five and five make ten. n/a	worked because there weren't as many dots? Why were those ones harder? Did you count them one by one, and that didn't work because there were more dots and you ran out of time? What's a faster way than counting? Could we use patterns like on the die? So these are patterns we know. We know when we see this pattern (hold up five) that it is five because every time we count these dots we get five and soon we won't have to count the dots because we just know that this pattern is five. It will always be five. Were they harder because we haven't seen them a lot? So if we knew that this pattern was five, if we have two of those patterns together, and we know that five plus five is ten, then our

	So today we found out how many objects we had by quickly seeing a pattern. When we immediately—or right away—know the number of objects just by seeing the pattern, we are using a different way of counting than when we count each object. You may use both ways of counting. Imagine you have a group of 9 snap cubes, and I ask you to tell me how many there are. Would it be faster to just see it like our patterns, or to count them one by one? Why? What if I had 16?	Just seeing it! Because by just seeing it I can tell you how many in five seconds or less, but if I have to count them it might take longer. Count them	five patterns put together is ten. Right, it might take longer to count them, especially if we accidentally count one twice, or miss one. We could, or what if I could see some patterns in my group, but then there are still other cubes left. Could I see those patterns and add them, and then count the rest of them one by one? For example, I see a group of five here and a group of five here, but these others are too scattered. I know

			that five and five is ten, so I'll start at ten, and then count these other ones. So I have ten, then 11, 12, 13, 14, 15, 16.

5. Assessment of Student Learning

Plan for assessment (both formative and summative) to assess student learning. Some guiding questions to consider:

- What questions are appropriate for my students to do after the investigation?
    - How many objects are here/do you see/do I have?
- What are the goals of the homework/**classwork** assignment?
    - For students to identify "how many?" are in a set using subitizing.
- How will students be supported in completing the assignment? Do I provide information and support for students and parents?
    - A station activity with teacher support for questions and greater thinking, discussions afterwards, and reviews beforehand.
    - Students will not have homework other than practicing identifying "how many?" of a group of objects.
    - Parents will get folders at the end of every week with the students work for that week, and newsletters once a month with a section on "what's coming up in math!"

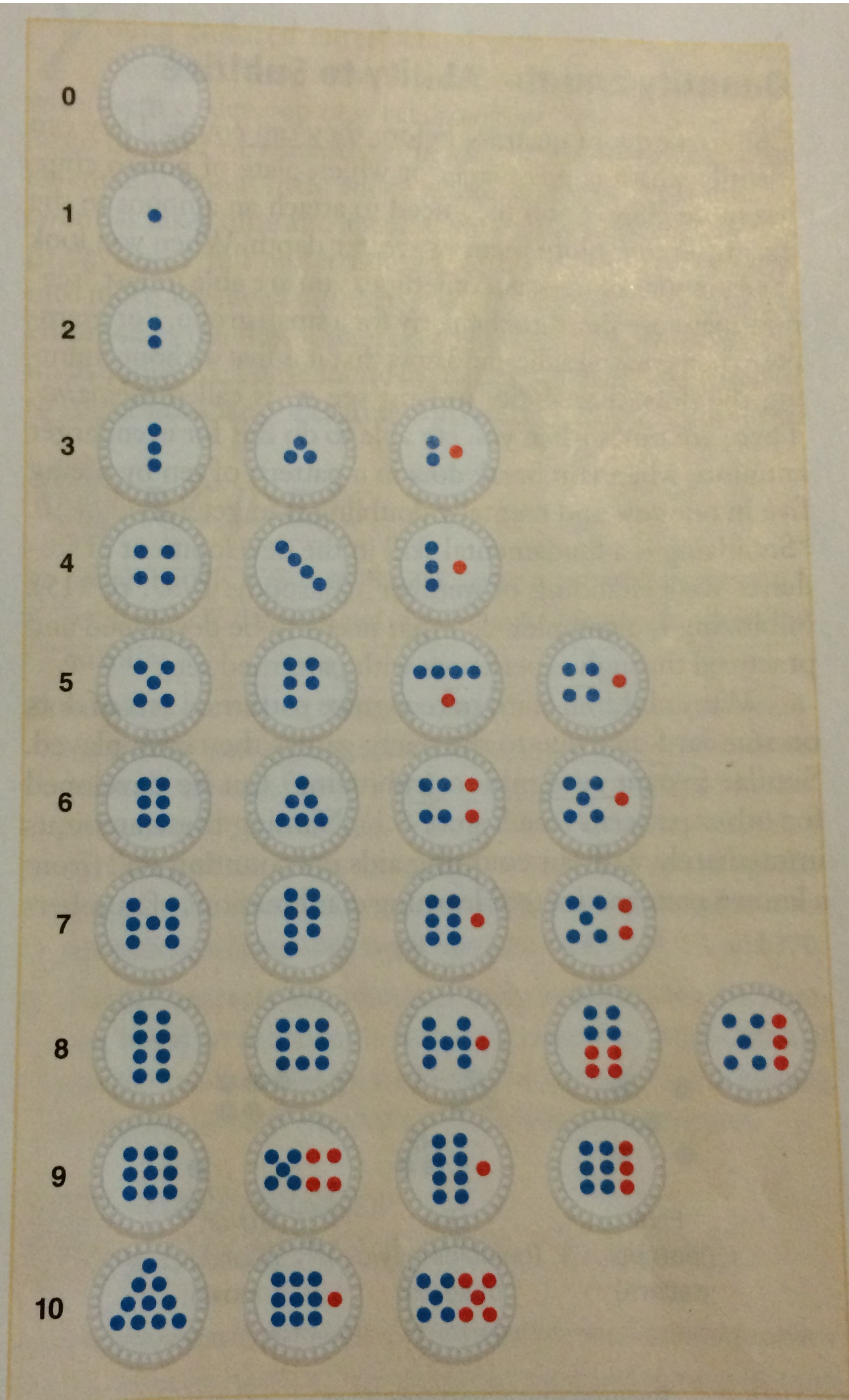


FIGURE 8.2 A collection of dot patterns for "dot plates."

APPENDIX B

Research-based Instructional Design Practices Specific to Mathematics

—Launch, Explore, Summarize Lesson Plan Format—

- Description of the Lesson
  - Subject: Math-Cardinality
  - Grade: Kindergarten
  - Timing (How long will the lesson take?): 35 minutes
  - Reference for lesson:
    - Sorting bottles and pom-poms: http://theimaginationtree.com/2013/03/sort-and-count-maths-bottles.html

4. Learning Goals and Objectives

  - What is the essential content of the instruction? (What are the big mathematical ideas of the investigation?)
    - Students will connect numbers with quantities.
    - Students will be able to count (recite the number sequence), giving one-to-one correspondence to objects.
  - CCSS Standard (What do I want kids to know when this investigation is finished?)

CCSS.MATH.CONTENT.K.CC.B.4
Understand the relationship between numbers and quantities; connect counting to cardinality.

CCSS.MATH.CONTENT.K.CC.B.4.A
When counting objects, say the number names in the standard order, pairing each object with one and only one number name and each number name with one and only one object.

CCSS.MATH.CONTENT.K.CC.B.5
Count to answer "how many?" questions about as many as 20 things arranged in a line, a rectangular array, or a circle, or as many as 10 things in a scattered configuration; given a number from 1-20, count out that many objects.

  - What are my specific and measurable objectives for the lesson?
    - Students will count the appropriate number of objects as the number on the bottle indicates.
    - Students will count by reciting the number sequence as they give one-to-one correspondence to each pom-pom as they put it in the bottle.
  - Consider student background, knowledge and experience. What prior knowledge do the students have concerning this topic?

    Students need to recognize the number(s) given their numeral, symbol, or word form.

Students need to have the fine motor skills to drop a pom-pom into a water bottle.

- ❖ What mathematical vocabulary does this investigation bring out?

  Number- a value, quantity, or amount expressed by a word, symbol, or figure

  Number word- the written term for the number (i.e. four)

  Number symbol- a symbolic representation of a number (tallies, dots, etc.)

  Numeral- the digital term for the number (i.e. 4)

  Manipulative terms: object, bottle, pom-pom

  Colors: red, blue, green, yellow, purple, orange

- ❖ What misconceptions might arise?

  Not recognizing the numeral form of the number, this is not necessarily a misconception, but a concept that needs to be taught/worked on.

  That they are sorting colors instead of counting objects.

- ❖ Curriculum adaptations and instructional modifications: (modifications to meet the needs of diverse populations and students with various ability levels)

  Have the word form of the number and worksheets in English and home language (Spanish, French, etc.)

  Larger images for visually impaired students, larger manipulatives for students with delayed fine motor skills.

5. Resources, Materials, and Preparation for Instruction (What facilities, resources, and tools will be needed and how are they to be used?)

   - Collaborators (Other teachers, outside visitors, or technical staff): assistant teacher, if there is one, to help monitor stations and assist students
   - Technological tools for the teacher (demonstration): n/a
   - Technological tools for the student: Smart Board
   - Manipulatives: enough clear water bottles for four per student; multi-colored pom-poms; dry erase markers—same colors as pom-poms, and a black; pencils
   - Handouts and recording sheets (include a copy): Pom-Pom Counting: How many?

6. Instructional Method and Procedures
   a. **Launch** (5 minutes)

Teacher Considerations (Before)	Description of Learning Activities	Anticipated Student Responses	Teacher Guidance (During)
• I will engage students' prior knowledge by having them count along with the Youtube video: https://www.youtube.com/watch?v=6RfIKqkvHTY	We're all going to count along with the video! Get your counting voices ready! I want to hear you loud and proud.	Student(s) count in standard order along with the video, loudly and clearly.	Positive reinforcement for those who are participating. Do the video with the students!

b. Explore (15 minutes)

Teacher Considerations (Before)	Description of Learning Activities	Anticipated Student Responses	Teacher Guidance (During)
Answer the following about planning this phase: • Each child has individual paper and sorting bottles, but the table group may share pom-poms. • Materials: Worksheets, pencils, bottles, pom-poms. • Using the empty water bottles, write, in black dry-erase, a number on each bottle. Then color a band around each bottle, give a different color to each one (this is for the worksheet). • Advantages: students engage in material in hands-on activities with the support of the teacher and teacher aid. Stations allow students to interact with the	Have students sort the pom-poms into each bottle, counting out loud as they do so, according to the number on the bottle. After you put the pom-poms in a bottle, write down how many pom-poms are in the bottle on your worksheet. For example, my worksheet has a red bottle here with a line next to it (show on worksheet). I'm going to write how many pom-poms I have in my red bottle on this line (show on worksheet). I'll do that for each color bottle on my sheet. If you finish, I'll come around and change the numbers on your bottles, and you'll write those answers down on these second lines here (point to them on worksheet).	Students use one-to-one correspondence to drop each pom-pom into the bottle, counting out loud, to fill it up with the correct number that is written on the bottle. They then record the number of pom-poms in the bottle on their worksheet. Some students may sort by color instead of paying attention to the numbers and counting the specified amount of pom-poms into the bottle. May have trouble recognizing the numeral.	If students finish quickly, come around with a black dry-erase marker and change the numbers so they can keep working. If students are sorting by color instead of paying attention to the numbers, redirect them by restating what they should be doing. If students are having trouble with recognizing the numeral, you can write it in word-form and/or in symbolic form in addition to the numeral form. You can also give them a hint for recognizing the numeral: Remember a four looks similar to a goal post or some other descriptor

material and content at their own level/pace. • Disadvantages: students focus on the fun and 'play' of the stations over the number connection. Students may sort by color of bottle and pom-poms instead of focusing on the numbers.		May put in the wrong amount of pom-poms.	that has previously been decided upon in class. Guide with questions: How many pom-poms did you put in? How many did the bottle say to put in? Can you figure out how to get to the number on the bottle?

c. Summarize (15 minutes)

Teacher Considerations (Before)	Description of Learning Activities	Anticipated Student Responses	Teacher Guidance (During)
Answer the following about planning this phase: • To assign students a number for discussion, you may do so randomly as you call on them OR you may have them draw a number out of a bowl. This latter option gives the students with higher numbers some time to think about how they will write and represent it. Provide some wait time for the early numbers to think about their answers as well.	• If you used the number out of a bowl: Ask who has number 1. Then have the student write (numeral and/or word) and represent the number symbolically on the Smart Board. • If you decide to call on students randomly and assign the number: ask the student to write (numeral and/or word) and represent the number symbolically on the Smart Board.	Correctly writing and representing the number. Writing the wrong numeral and/or misrepresenting the number.	Positive Reinforcement, also ask them the following questions: What number did you have? And what did write? And how many did you draw? This helps them reaffirm their answer, and it sets up the classroom atmosphere so that when you ask these for the student you know are incorrect, they don't feel as if you are targeting them. Guide to answer: What number did you have on your paper? And what number did you write? And how many did you draw? The student should, hopefully, recognize their mistake and correct himself or herself. If not, guide them more specifically, pointing them to the number somewhere on the wall if it's a problem with the numeral, or helping them count

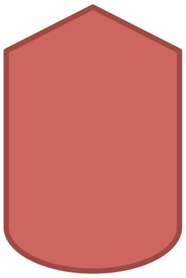
			to get to the correct symbolic representation—if they drew too many, suggest via question to erase the extra, if they drew too few, suggest via question drawing the rest.
		Students may flip the numeral in some way (backwards or upside down), or if writing the word form, they may spell it incorrectly.	If they flip their numeral or spell it wrong, direct them to check their numeral/spelling with the numbers on the wall.

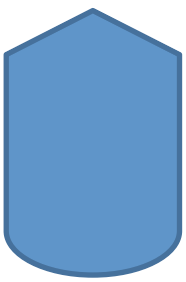
5. Assessment of Student Learning
Plan for assessment (both formative and summative) to assess student learning. Some guiding questions to consider:

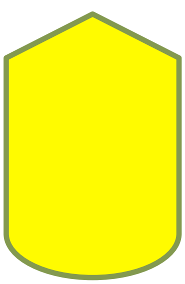
- What questions are appropriate for my students to do after the investigation?
    - The summarize is also the assessment.
        a. Can you write in numeral form a given number?
        b. Can you write in word form a given number?
        c. Can you represent symbolically (draw out) a given number?
- What are the goals of the homework/**classwork** assignment?
    - For students to correctly write and represent numbers
- How will students be supported in completing the assignment? Do I provide information and support for students and parents?
    - Activities completed with teacher support for questions and deeper thinking; discussions afterwards and reviews beforehand also promote student understanding.
    - Students will not have directly assigned homework, but students/parents will be encouraged to practice reciting the counting sequence with one-to-one correspondence to objects at home with their parents/students.
    - Parents will get folders at the end of every week with the students work for that week, and newsletters once a month with a “what’s coming up in (subject)!”

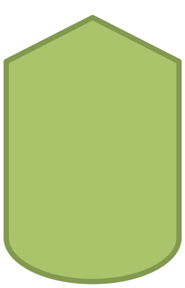
Name: ____________________

Pom-Pom Numbers









APPENDIX C

Research-based Instructional Design Practices Specific to Mathematics

—Launch, Explore, Summarize Lesson Plan Format—

1. Description of the Lesson
    - Subject: Math-Cardinality
    - Grade: Kindergarten
    - Timing (How long will the lesson take?): 45 minutes
    - Reference for lesson: Clements and Sarama (2014): learning trajectory for counting—instructional activities for Counter (small numbers)
2. Learning Goals and Objectives
    - What is the essential content of the instruction? (What are the big mathematical ideas of the investigation?)
        - Students will use one-to-one correspondence and/or subitizing to determine cardinality and produce an amount.
    - CCSS Standard (What do I want kids to know when this investigation is finished?)
        - CCSS.MATH.CONTENT.K.CC.B.4 Understand the relationship between numbers and quantities; connect counting to cardinality.
        - CCSS.MATH.CONTENT.K.CC.B.4.B Understand that the last number name said tells the number of objects counted. The number of objects is the same regardless of their arrangement or the order in which they were counted.
        - CCSS.MATH.CONTENT.K.CC.B.4.C Understand that each successive number name refers to a quantity that is one larger.
        - CCSS.MATH.CONTENT.K.CC.B.5 Count to answer "how many?" questions about as many as 20 things arranged in a line, a rectangular array, or a circle, or as many as 10 things in a scattered configuration; **given a number from 1-20, count out that many objects**.
        - From thesis, pg. 11, paragraph 2: Instantly recognize known patterns up to ten in a variety of arrangements to answer "how many."
    - What are my specific and measurable objectives for the lesson?
        - Students will answer "how many" using one-to-one correspondence or subitizing.
        - Students will produce a set of objects of up to 10.
    - Consider student background, knowledge and experience. What prior knowledge do the students have concerning this topic?
        - Students need to know their numbers up to 10
    - What mathematical vocabulary does this investigation bring out?
        - Number- a value, quantity, or amount expressed by a word, symbol, or figure
        - Number symbol- a symbolic representation of a number (tallies, dots, etc.)
        - Numeral- the digital term for the number (i.e. 4)
        - Manipulative terms: subitizing flash cards, game board, game piece
    - What misconceptions might arise?
        - Counting a symbol more than once if using one-to-one correspondence.

    - Moving your game piece an incorrect amount due to miscounting
- Curriculum adaptations and instructional modifications: (modifications to meet the needs of diverse populations and students with various ability levels)
    - Larger images for visually impaired students
    - Larger manipulatives for students with delayed fine motor skills.

3. Resources, Materials, and Preparation for Instruction (What facilities, resources, and tools will be needed and how are they to be used?)

- Collaborators (Other teachers, outside visitors, or technical staff):
    - Assistant teacher if have one to help monitor stations and assist students
- Technological tools for the teacher (demonstration):
    - N/A
- Technological tools for the student:
    - N/A
- Manipulatives:
    - Game board, game pawns, subitizing flash cards
- Handouts and recording sheets (include a copy):
    - Subitizing flash cards, game board

4. Instructional Method and Procedures
    a. **Launch** (10 minutes)

Teacher Considerations (Before)	Description of Learning Activities	Anticipated Student Responses	Teacher Guidance (During)
I will engage students' prior knowledge by having them count along with the Youtube video: https://www.youtube.com/watch?v=6RflKqkvHTY	We're all going to count along with the video! Get your counting voices ready! I want to hear you loud and proud.	Student(s) count in standard order along with the video, loudly and clearly.	Positive reinforcement for those who are participating. Do the video with the students!
I will also engage students' prior knowledge discussing board games	Who has played a board game before? If you have played a board game before I want you to put a thumbs up on your chest.	Students may or may not put a "thumbs up" on their chest, indicating whether or not they have played a board game before.	Okay, I see some thumbs which is good. Can someone with a thumbs up—raise your hand please—explain to us how you play a board game?
Lastly I will introduce	Who can tell me how you play a typical	You draw a card or sometimes you roll a	Good! Do those of you who have played before

the subitizing flash cards.	board game? I have some cards, here (show them). They have dots on them and you will need to figure out how many dots are on the card. You can do this by counting each dot, or you might look at it and just know what the number is like we did with the plates a couple days ago. You will use these cards in a little bit.	die, and then you move your person that many spaces.	agree with that? Do those of you who have not played a board game before understand how you play?—if not have a student repeat how you play.

b. Explore (25 minutes)

Teacher Considerations (Before)	Description of Learning Activities	Anticipated Student Responses	Teacher Guidance (During)
● Supplies: boards for the board game, game pawns, subitizing flash cards. Make sure you have enough for groups of 3-4. ● Advantage: students engage in material in hands-on activities with the support of the teacher and teacher aid. ● Advantage:	Today we are going to play a board game! You will be in table groups to play the board game. Listen carefully so you know how to play: you will take turns! When it is your turn, draw a card—show the subitizing flash cards—and figure out how many dots are on the card. Then you will move your game piece that many spaces on the game board. When	As students are playing: they can figure out how many and move that many spaces As students are playing: they may make a mistake figuring out how many or moving that many spaces As students are playing: being a good sport or playing fair	Provide positive reinforcement to students. Allow teammates to try to help first, then provide a helpful hint or reminder that can help the student when figuring out the number: such as you can finger point to each object and count it. If they are being good, provide positive reinforcement. If an

allows students to interact with the material and content at their own level/pace. ● Disadvantage: Students may focus on winning or losing, which could cause a distraction. Try to prevent this by going over “fair play” and “good sport” rules. ● Oral directions to get students to their seats if they are not in them: When I call your team I want you to go to your table. Do not touch the materials until I say: ‘GO.’ [Repeat until students are all at tables] ● Oral directions if students are in their seats already, make sure to tell them “when I say Go…” then give directions. That way they aren’t playing with the materials before they are supposed to.	everyone makes it to the end, start over. if you understand, put a thumbs up. If you need help, who could you ask? If your teammates notice that you have moved too many spaces, or too little, and they tell you, should you get upset? Should you say thank you and correct your piece? Teammates, should you yell at the person who made a mistake or should you be nice about it? Does it matter who wins or loses? Send students to their tables to play.	may or may not be an issue Students make a thumbs up or they do not Teammates, teacher No, you shouldn’t. Yes, you should change our piece to the right spot and say thanks. No, yelling at friends is mean. No it doesn’t matter who wins or loses. Or they may say it does.	issue arises, address it with that group, if one student is repeatedly making a scene, pull them out of the game and explain being a good sport. Right You should say I don’t think you moved that right let’s double check. Are we playing to help us learn how to answer “how many?” If this is why we are playing, then does it matter who wins? If you can figure out how many and move that many spaces, you are a winner.

c. Summarize (10 minutes)

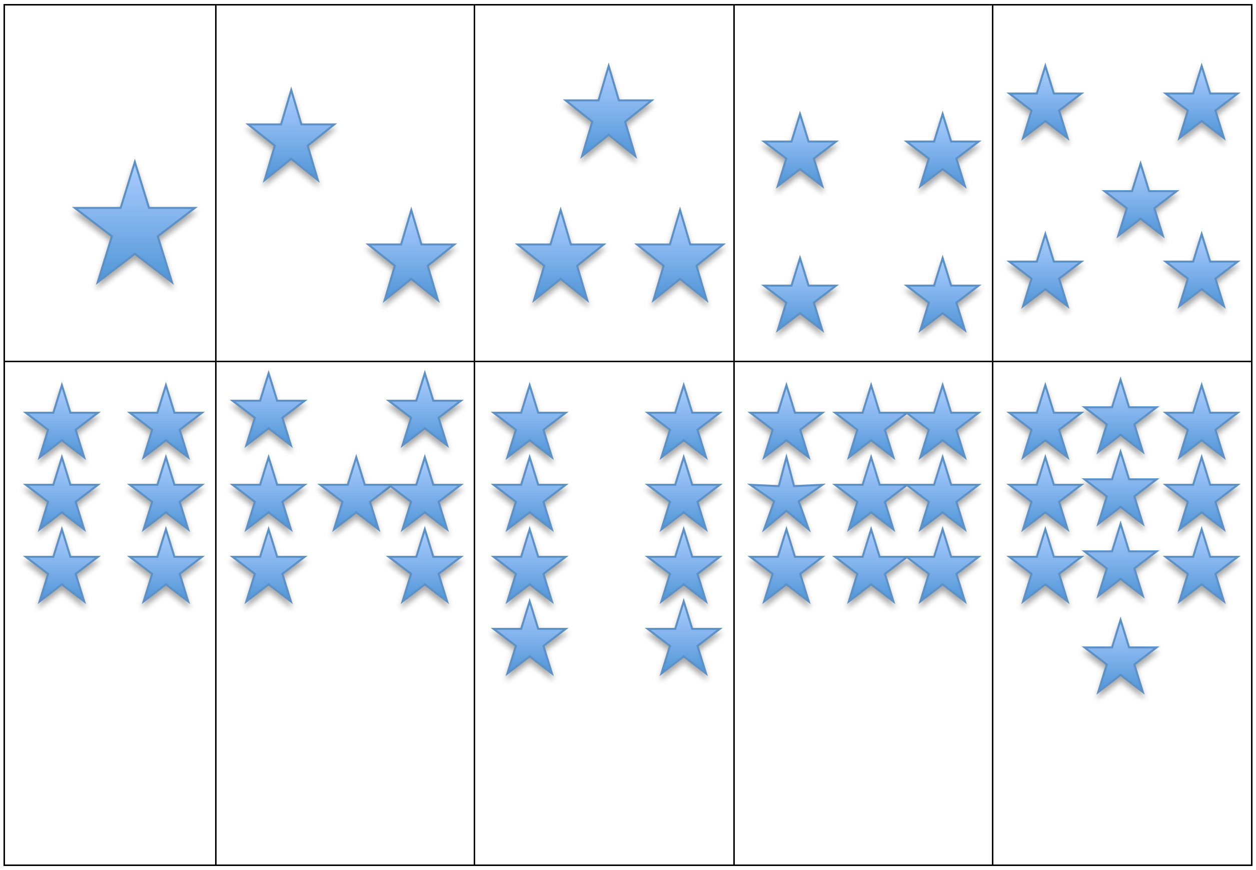
Teacher Considerations (Before)	Description of Learning Activities	Anticipated Student Responses	Teacher Guidance (During)
Bring students over to the whole group area so that they are away from the manipulatives.	Call students over to the whole group area by table.	Students come to the area.	
Assess students on how well they played the game as far as being a good sport goes and whether or not they stayed on task.	Thumbs up if you were a good sport. Thumbs down if you were not and had to be talked to. Thumbs sideways if I talked to your group, but you did better after that.	Students put a thumbs up, down, or sideways.	Remind students how they can get a thumbs up and what a thumbs down or sideways looks like.
	Thumbs up if you stayed on task and continued playing the game even after someone had made it to the end. Thumbs down if you didn't. thumbs sideways if you need work.	Students put a thumbs up, down, or sideways	Remind students how they can get a thumbs up and what a thumbs down or sideways looks like.
Major concepts to address: how did they figure out how many? One-to-one or subitizing-which one helped more. Were they able to produce this amount by moving their pawn the right amount of spaces?	How did you figure out how many spaces you needed to move? Turn and Talk with a partner. (provide ample wait time)	Students may say: we drew a card and figured out how many dots were on it.	Have some students share what they talked about or repeat some of the answers you heard.
	How did you figure out how many dots were on the card? (call on a student, then have someone repeat the answer, then have someone	Students may say: I counted each one. Or they may say I just knew what the pattern was. To add on, a student may add the answer that was not	Have some students share what they talked about or repeat some of the answers you heard. Expand: Counting them is one way you could figure

	add on to the conversation)	said, or they may explain what they mean by "I just knew the pattern"	it out. How else could you figure it out? What does just knowing the pattern mean? Were there some patterns that were easier for you to recognize? Did both ways help you answer 'How many?" Are they both useful?
	Was it faster to recognize the pattern—the fancy word that teachers use for this is subitizing- so was it faster to recognize the pattern of dots or to count each one? Why?	Students may say: yes, because they were small numbers and if I knew the pattern I could say it real quick but if I had to count it, it took me a little longer. I still had to count the ones that were bigger numbers because I didn't know those patterns yet.	So if the number is easy to recognize, it may be faster for us to just see how many there are without counting. Sometimes though we have really big numbers.
	What if my number was high, like 25? (show the pattern for the number on a dot plate or piece of paper or a Smart Board if your classroom is equipped with one) Who would count that? Who would try to recognize some patterns in the number? How could I recognize that pattern?	Students may say: I would have to count that, it's so big! Or they may say they can figure out the pattern. Well I see a five there and ten there and another ten there. But I don't know how that makes 25 (depending on if they can add yet or not).	So sometimes we might just have to count for right now. Or if we see a pattern we know, we can start from that number and keep counting. Or, if you see multiple patterns you recognize, like the 5, ten, and ten, you would have to add those numbers together; we'll get there later in the year (if they are not already there).

5. **Assessment of Student Learning**
   Plan for assessment (both formative and summative) to assess student learning. Some guiding questions to consider:

   - What questions are appropriate for my students to do after the investigation?
     - How many dots are on this card?
     - How did you figure out how many dots were there?
     - How many spaces do you need to move?
     - How many more spaces do you need to get to the (insert color or to the end)?
   - What are the goals of the homework/**classwork** assignment?
     - For students to correctly identify "how many" and to produce the number identified by moving that many spaces.
   - How will students be supported in completing the assignment? Do I provide information and support for students and parents?
     - The lesson allows for teacher support to answer questions and develop deeper thinking.
     - Students will not have homework besides practicing answering "how many?"
     - Parents will get folders at the end of every week with the students work for that week, and newsletters once a month with a section about "what's coming up in math!"

Subitizing Flash Cards



Game Pieces: May be any object you like that you have that are multi colored so each player is a different color (e.g.-counting bears, pattern blocks, pom-poms, or if you have spare game pawns lying around your house, you can use those too!)

Game Board: